

# Transverse Field Ising Model Under Hyperbolic Deformation

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Ground state of the one-dimensional transverse field Ising model is investigated under the hyperbolic deformation, where the energy scale of  $j$ -th bond is proportional to the function  $\cosh[j\lambda]$  that contains a parameter  $\lambda$ . Although the Hamiltonian is position dependent, the ground state is nearly uniform and finitely correlated. We observe the energy cross over between the ordered and disordered state with respect to the transverse field. The model shows first order phase transition, and the discontinuities in the magnetization and entanglement entropy at the transition point detect the Ising universality.

KEYWORDS: DMRG, Hyperbolic, Transverse Field, Ising, Entanglement

## 1. Introduction

Spacial uniformity is a fundamental concept in physics. If a Hamiltonian of a system is uniform, in the manner that it is represented as a spacial sum or integral of position independent local terms, the corresponding ground state is expected to be uniform, provided that there is no spontaneous symmetry breaking that causes spacial modulation. How about the opposite? When the ground state is uniform, is it expected that the Hamiltonian is also uniform? This is not true; a nonuniform Hamiltonian can possess a uniform ground state. A trivial example is the system that consists of independent spins. Consider the position dependent spin Hamiltonian

$$H = -\Gamma \sum_j g_j \sigma_j^x, \quad (1.1)$$

where  $\Gamma > 0$  represents the external magnetic field to the  $x$ -direction,  $g_j$  a site dependent positive factor at  $j$ -th site, and  $\sigma_j^x$  the Pauli operator. The corresponding ground state is the complete ferromagnetic state, where all the spins are polarized to the  $x$ -direction.

The example in Eq. (1.1) might be too trivial, since there is no inter-site couplings. Thus let us consider the transverse field Ising (TFI) model on the one-dimensional (1D) lattice, as a more realistic reference system. We treat the position dependent TFI model defined by the Hamiltonian

$$H = -J \sum_j f_j \sigma_j^z \sigma_{j+1}^z - \Gamma \sum_j g_j \sigma_j^x, \quad (1.2)$$

where  $J > 0$  represents the longitudinal nearest-neighbor coupling, and  $f_j$  is a site dependent positive factor. When all the factors  $f_j$  and  $g_j$  are equal to unity, the ground state of this system is uniform in the thermodynamic limit. In this uniform case there is a quantum phase transition at  $\Gamma = J$ , where the model becomes self dual.<sup>1)</sup> (See appendix.)

In this article we focus on the case where the position dependence is given by  $f_j = \cosh[j\lambda]$  and  $g_j =$

$\cosh[(j - \frac{1}{2})\lambda]$ , where  $\lambda$  is a nonnegative parameter. The corresponding Hamiltonian

$$H^c(\lambda) = -J \sum_j \cosh[j\lambda] \sigma_j^z \sigma_{j+1}^z - \Gamma \sum_j \cosh[(j - \frac{1}{2})\lambda] \sigma_j^x \quad (1.3)$$

for the case  $\lambda > 0$  can be interpreted as the one parameter deformation — the *hyperbolic deformation* — to the uniform case when  $\lambda = 0$ . The ‘bulk’ part of the ground state of  $H^c(\lambda)$  is expected to be uniform even when  $\lambda > 0$ , since the path-integral representation of the imaginary time evolution by  $H^c(\lambda)$  would be given by uniform classical action on the  $1 + 1$  hyperbolic plane.<sup>2)</sup> Such a uniformity under hyperbolic deformation has been observed for the deformed  $S = 1/2$  and  $S = 1$  Heisenberg spin chains.<sup>3,4)</sup> We confirm this uniformity for the case of the deformed TFI model in the next section.

We employ the density matrix renormalization group (DMRG) method,<sup>5–8)</sup> and obtain the lowest energy states of finite size systems under both ferromagnetic and paramagnetic boundary conditions. A natural interest in the ground state of  $H^c(\lambda)$  is the ordered-disordered transition. A classical analogue of the deformed TFI model is the classical Ising model on the two-dimensional (2D) hyperbolic lattices,<sup>9)</sup> which exhibit the mean-field like second-order phase transition.<sup>10–21)</sup> How about the deformed TFI model? We investigate the spontaneous magnetization in §3. In contrast to the classical cases, the deformed TFI model exhibits first-order transition. The  $\lambda$ -dependence of the quantum entropy is also observed. Conclusions are summarized in the last section.

## 2. Uniformity in the Ground State

Under the hyperbolic deformation, the energy scale in the Hamiltonian  $H^c(\lambda)$  blows up exponentially with  $|j|$ . In order to well define the eigenvalue problem for  $H^c(\lambda)$ , we consider finite-size systems. For simplicity, we treat the cases where system size  $L$  is even, and label sites from

$j = -L/2 + 1$  to  $j = L/2$ . Therefore the center of the system is between  $j = 0$  and  $j = 1$ , where the strength of the nearest neighbor interaction is the smallest.

Boundary condition is essential for the determination of the ground state, since the ratio of the boundary (or surface) energy with respect to the total energy does not vanish in the large  $L$  limit.<sup>11,12)</sup> We have to aware of this characteristic behavior of the hyperbolic deformation when we consider the ground-state phase transition. We choose either the paramagnetic boundary condition or the ferromagnetic one. The former is imposed by fixing the spins of both ends to the  $x$ -direction. Since the inter-site coupling in the TFI model is mediated only by longitudinal interaction  $\sigma_j^z \sigma_{j+1}^z$ , the paramagnetic boundary condition decouples the boundary sites at the both ends  $j = -L/2 + 1$  and  $j = L/2$  from the inner part  $-L/2 + 2 \leq j \leq L/2 - 1$ . As a result, the effect of paramagnetic boundary condition is represented by the Hamiltonian

$$H_{P;L}^c(\lambda) = -J \sum_{j=-L/2+2}^{j=L/2-2} \cosh[j\lambda] \sigma_j^z \sigma_{j+1}^z \quad (2.1)$$

$$- \Gamma \sum_{j=-L/2+2}^{j=L/2-1} \cosh[(j - \frac{1}{2})\lambda] \sigma_j^x,$$

where there is no constraint for all the spins that appear in the above equation. On the other hand, the ferromagnetic boundary condition is imposed by fixing the boundary sites to the  $z$ -direction. In this case the Hamiltonian is expressed as

$$H_{F;L}^c(\lambda) = H_{P;L}^c(\lambda) + \cosh[(\frac{L}{2} - 1)\lambda] (\sigma_{-L/2+2}^z + \sigma_{L/2-1}^z), \quad (2.2)$$

which is equivalent to put magnetic field to  $z$ -direction only at the position  $j = -L/2 + 2$  and  $j = L/2 - 1$ . Thus under both boundary conditions we effectively treat  $(L - 2)$ -site system in numerical analysis. We choose the parameter  $J$  as the unit of energy throughout this article.

We employ DMRG method<sup>5-8)</sup> for the numerical determination of the ground state. A direct application of the finite-system DMRG algorithm encounters a numerical instability, which is caused by the blow-up of the energy scale  $\cosh[j\lambda]$  with respect to  $|j|$ . In order to stabilize the computation, we treat

$$\tilde{H}_{P;L}^c(\lambda) \equiv H_{P;L}^c(\lambda) - \langle H_{P;L}^c(\lambda) \rangle$$

$$\tilde{H}_{F;L}^c(\lambda) \equiv H_{F;L}^c(\lambda) - \langle H_{F;L}^c(\lambda) \rangle \quad (2.3)$$

instead of  $H_{P;L}^c(\lambda)$  and  $H_{F;L}^c(\lambda)$  directly, where  $\langle \rangle$  denotes expectation value taken by the ground state. The smallest eigenvalue of both  $\tilde{H}_{P;L}^c(\lambda)$  and  $\tilde{H}_{F;L}^c(\lambda)$  is zero by definition. Since the ground state is not a priori known, the subtraction process in Eq. (2.3) is performed iteratively. The decay of the density matrix eigenvalue is rapid,<sup>21)</sup> and for all the range of deformation parameter we examined  $1/32 \leq \lambda \leq 1$  it is sufficient to keep  $m = 16$  block-spin states; even for the worst case  $\lambda = 1/32$  the truncation error is of the order of  $10^{-10}$  in the density matrix eigenvalues.

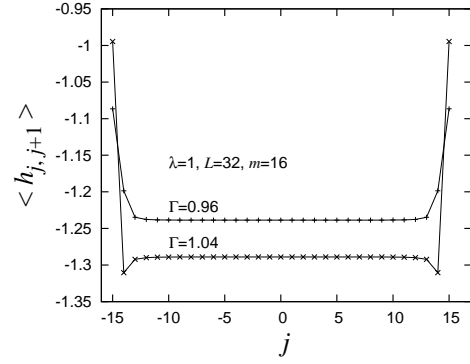


Fig. 1. Expectation value of  $h_{j,j+1}$  in Eq. (2.4) for the ground state when  $\Gamma = 0.96$  and  $\Gamma = 1.04$ .

For the analysis of local energy, we employ a bond operator

$$h_{j,j+1} = - \left[ \frac{\Gamma}{2} \sigma_j^x + J \sigma_j^z \sigma_{j+1}^z + \frac{\Gamma}{2} \sigma_{j+1}^x \right], \quad (2.4)$$

which coincides with the local energy of the  $j$ -th bond of the undeformed TFI model. Figure 1 shows the expectation value  $\langle h_{j,j+1} \rangle$  calculated for the ground state when  $\lambda = 1$  and  $L = 32$ . We choose two typical cases,  $\Gamma = 0.96$  with ferromagnetic boundary condition and  $\Gamma = 1.04$  with paramagnetic one. Despite of the very strong position dependence in the Hamiltonian, boundary correction decays rapidly, and  $\langle h_{j,j+1} \rangle$  is nearly uniform deep inside the system. Figure 2 shows the on-site transverse interaction  $\langle \Gamma \sigma_j^x \rangle$ , which is nearly uniform inside. We also observed similar uniformity for other local observables such as  $\langle \sigma_j^z \rangle$  and  $\langle \sigma_j^z \sigma_{j+1}^z \rangle$ . The rapid decay of the boundary corrections is observed for all the values of the transverse field  $\Gamma$  we have examined. The behavior suggests that the ground state of the hyperbolically deformed TFI model is always finitely correlated, and that the bulk part of the ground state is well approximated by the uniform matrix product.<sup>22,23)</sup>

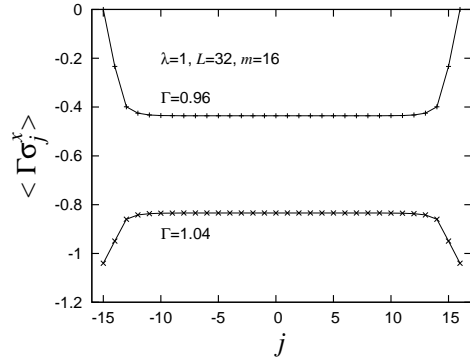


Fig. 2. On-site transverse interaction  $\langle \Gamma \sigma_j^x \rangle$ .

### 3. Phase Transition

In order to capture the nature of ground-state phase transition, let us compare the energy of the ferromagnetic state with that of paramagnetic one. It is, however,

difficult to directly compare  $\langle H_{F;L}^c(\lambda) \rangle$  with  $\langle H_{P;L}^c(\lambda) \rangle$ , since subtraction of the boundary energy is not straightforward. Therefore we use the expectation value  $\langle h_{0,1} \rangle$  at the center of the system as a representative value for the local energy density of the bulk. The uniformity observed in the previous section would justify this way of evaluation of the ground-state energy. Figure 3 shows  $\langle h_{0,1} \rangle$  for  $L = 8, 16, 24$ , and  $32$  when  $\lambda = 1$ . Solid lines and dotted lines, respectively, represent data calculated with ferromagnetic and paramagnetic boundary conditions. System size dependence between  $L = 24$  and  $32$  is negligible, and for these cases the solid and dotted lines cross at  $\Gamma = 1$ . Although the deformed Hamiltonian is not invariant under the duality transformation, still the lowest energy state alternates at the point  $\Gamma = 1$ . (See Appendix.)

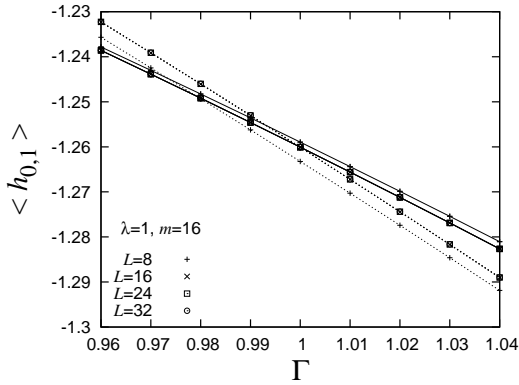


Fig. 3. Energy density  $\langle h_{0,1} \rangle$  at the center of the system. Solid lines and dotted lines, respectively, represent data calculated with ferromagnetic and paramagnetic boundary conditions. The  $L$ -dependence is very small for  $L \geq 16$ .

It should be noted that in a certain region of  $\Gamma$  around  $\Gamma = 1$  the ferromagnetic and the paramagnetic states coexist even in the large  $L$  limit, and one of these states is chosen by the imposed boundary condition. Thus one should be careful about the definition of the phase transition. Since we have considered that  $\langle h_{0,1} \rangle$  represents the bulk property of the system, it is natural to choose the state that gives lowest value of  $\langle h_{0,1} \rangle$  as the ‘stable’ ground state. The ground-state energy thus determined has a kink at the transition point  $\Gamma = 1$ . We checked the presence of kink in  $\langle h_{0,1} \rangle$  for the cases of weaker deformation down to  $\lambda = 1/32$ , where the sufficient system size  $L$  to obtain the convergence in  $\langle h_{0,1} \rangle$  grows roughly proportional to  $1/\lambda$ . The fact suggests that the hyperbolic deformation introduces a length scale  $\xi \propto 1/\lambda$  to the system, which coincides the radius of curvature  $R \propto -1/\lambda$  of the corresponding hyperbolic plane.<sup>2,9)</sup> Actually, the length  $\xi$  already appeared as the dumping length in Fig. 2.

Observation of the order parameter clarifies the nature of phase transition. Figure 4 shows the magnetization  $\langle \sigma_0^z \rangle$  at the center of the system  $j = 0$ , plotted for the states that gives lowest value of  $\langle h_{0,1} \rangle$ ; the ferromagnetic boundary condition is imposed when  $\Gamma \leq 1$ , and the paramagnetic condition when  $\Gamma \geq 1$ . We show

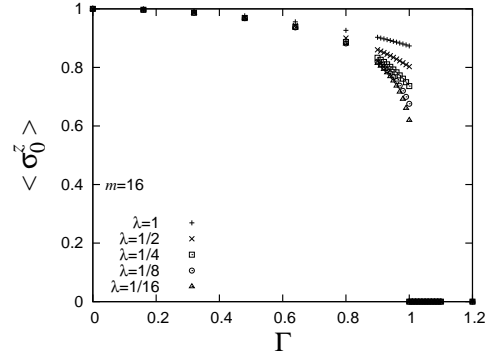


Fig. 4. Order parameter  $\langle \sigma_0^z \rangle$  observed at the center of the system.

the results when  $\lambda = 1, 1/2, 1/4, 1/8$ , and  $1/16$  calculated for sufficiently large system size  $L$ . There is a finite jump in  $\langle \sigma_0^z \rangle$  at the transition point  $\Gamma = 1$ , and thus the phase transition is first order. The  $\lambda$  dependence of this jump is plotted in Fig. 5. The eighth power of the jump is proportional to  $\lambda$ . This dependence is consistent with the Ising universality. The length scale  $\xi \propto 1/\lambda$  introduces an effective deviation to the transverse field of the amount

$$\Delta\Gamma \propto \xi^{-\nu} = \frac{1}{\xi} \propto \lambda \quad (3.1)$$

from the criticality of the uniform TFI model, where  $\nu = 1$  is the critical exponent for the correlation length. The observed jump in the spontaneous magnetization is therefore proportional to  $(\Delta\Gamma)^{1/8} \propto \lambda^{1/8}$ .

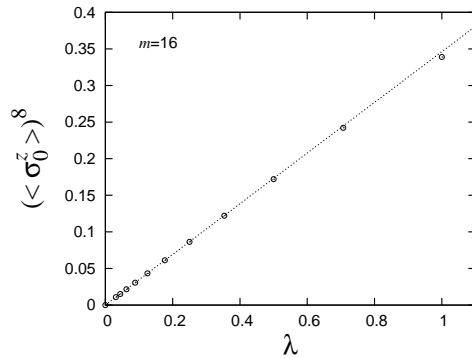


Fig. 5. Jump in the magnetization at  $\Gamma = 1$  with respect to  $\lambda$ .

The bipartite entanglement entropy  $S$  provides supplemental information for the phase transition. Figure 6 shows the entropy measured at the center of the system. Consistent with the spontaneous magnetization, the entropy does not show singular behavior around the transition point. Figure 7 shows the entropy at  $\Gamma = 1$  for the ordered state (A) and that for the disordered state (B), and also their sum (A)+(B) and difference (B)-(A). We draw a fitting line  $-(1/6) \log \lambda + 0.2475$  for the sum (A)+(B). This suggests that the average of the entangle-

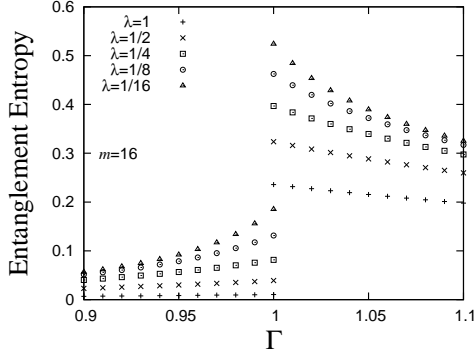


Fig. 6. Bipartite entanglement entropy measured at the center of the system.

ment entropy  $[(A)+(B)]/2$  at  $\Gamma = 1$  is expressed as

$$\frac{1}{2} \cdot \frac{1}{6} \log \frac{1}{\lambda} + \frac{0.2475}{2}, \quad (3.2)$$

for a wide range of  $\lambda$ . This dependence coincides with the fact that the leading term of the entanglement entropy is expressed as  $(c/6) \log W$ , where  $W$  is the system size and  $c$  is the central charge.<sup>24)</sup> This is because  $1/\lambda$  is proportional to the length scale  $\xi$ , and the TFI model belongs to the class where  $c = 1/2$ .

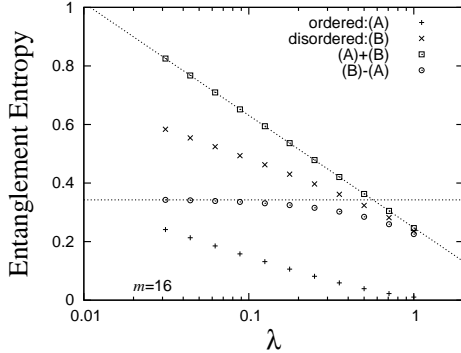


Fig. 7. Entanglement entropy with respect to  $\lambda$ .

The fitting to the difference (B)-(A) gives the estimate 0.3426 in the small  $\lambda$  limit. This value is about the half of  $\log 2 \sim 0.6931$ . The result suggests that difference of entanglement entropy between ordered and disordered states is a constant, if the correlation lengths of the both states are the same. We conjecture that the difference  $\sim (1/2) \log 2$  captures the boundary condition, but the detail is not clarified yet.

#### 4. Conclusion and Discussion

We have introduced the hyperbolic deformation to the 1D TFI model. It is shown that inner part of the ground state is uniform, as was observed other systems under hyperbolic deformation.<sup>2-4)</sup> The deformation introduces a characteristic length  $\xi$ , which is proportional to  $1/\lambda$ , and the ground-state phase transition becomes 1st order. Discontinuities in both spontaneous magnetization and bipartite entanglement entropy are consistent with the

Ising universality class.

The observed 1st order transition is different from the mean-field like 2nd order transition observed for the classical Ising model on hyperbolic lattices.<sup>10-21)</sup> Thus spatial anisotropy may change the nature of the phase transition. To clarify the difference, we have to construct a well defined quantum-classical correspondence by some means such as the Trotter decomposition.<sup>25, 26)</sup>

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#### Appendix: Duality Relation

The undeformed TFI model has a symmetry mediated by the duality relation.<sup>1)</sup> Let us introduce a new set of Pauli operators  $\tau_j^x$ ,  $\tau_j^y$ , and  $\tau_j^z$ . Then the Pauli operators  $\sigma_j^x$ ,  $\sigma_j^y$ , and  $\sigma_j^z$  can be expressed as

$$\begin{aligned} \sigma_j^x &= \tau_{j-1}^z \tau_j^z \\ \sigma_j^y &= - \left( \prod_{\ell=j-1}^{j-2} \tau_{\ell}^x \right) \tau_{j-1}^y \tau_j^z \\ \sigma_j^z &= \prod_{\ell=j-1}^{j-1} \tau_{\ell}^x, \end{aligned} \quad (A.1)$$

where the transformation is nonlocal. One can verify the relation  $\sigma_j^z \sigma_{j+1}^z = \tau_j^x$ . Substituting the above relation to  $H^c(\lambda)$  in Eq. (1.3), we obtain the Hamiltonian

$$\begin{aligned} H^c(\lambda) &= -J \sum_j \cosh[j\lambda] \tau_j^z \\ &\quad - \Gamma \sum_j \cosh[(j - \frac{1}{2})\lambda] \tau_{j-1}^x \tau_j^z \end{aligned} \quad (A.2)$$

written by the new set of Pauli operators. When  $\lambda = 0$  the transformed Hamiltonian has the same form as the original Hamiltonian, where parameters  $J$  and  $\Gamma$  are exchanged. For finite  $\lambda$ , the transformed Hamiltonian in Eq. (A.2) has the form where lattice indices in the deformation function is shifted by  $1/2$ . The observed energy crossover at  $\Gamma = 1$  in Fig. 3 suggest that this shift is not essential for the bulk part of the ground state. If we shift the index by  $1/4$  in advance and define the Hamiltonian as

$$\begin{aligned} H^{c'}(\lambda) &= -J \sum_j \cosh[(j + \frac{1}{4})\lambda] \sigma_j^z \sigma_{j+1}^z \\ &\quad - \Gamma \sum_j [\cosh(j - \frac{1}{4})\lambda] \sigma_j^x, \end{aligned} \quad (A.3)$$

then the transformed Hamiltonian is given by

$$H^{c'}(\lambda) = -J \sum_j \cosh[(j + \frac{1}{4})\lambda] \tau_j^z$$

$$-\Gamma \sum_j \cosh \left[ \left( j - \frac{1}{4} \right) \lambda \right] \tau_{j-1}^x \tau_j^z. \quad (\text{A}\cdot 4)$$

One finds the self duality between Eq. (A·3) and Eq. (A·4) when  $J = \Gamma$  even when  $\lambda$  is positive.

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